HE 215 : Nuclear & Particle Physics Course

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Outline I

- Relativistic Kinematics
  - Lorentz Transformations
  - Four-Vectors
  - Relativity Examples and Applications

Relativistic Kinematics

This is chapter 3 in Griffiths.

Relativistic Kinematics

Principle of Relativity

The laws of physics are the same in all inertial frames

Special relativity is crucial to understand the behaviour of fast-moving objects. In particle physics we are very often working with fast-moving objects.

Events and Observers

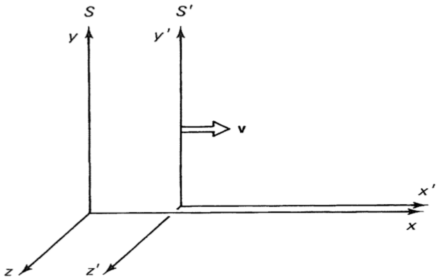
An event is any physical occurence we can say happened at a particular place and time. The place and time are specific to a given reference frame but the event does not belong to any specific frame, it happens in all frames.

An event is viewed by an observer in a specific reference frame.

It takes 4 numbers to quantify when and where an event occurs: 1 for time and 3 for space. These numbers are called the spacetime coordinates of an event.

Lorentz Transformations - Position

Event occurs at position (x, y, z) and time t is reference frame S. Same event space-time co-ordinates in S':



$$
\begin{aligned}
x' &= \gamma(x - vt) \\
y' &= y \\
z' &= z \\
t' &= \gamma\left(t - \frac{vx}{c^2}\right)
\end{aligned}
$$

$$\textit{with} \;\; \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Lorentz Transformations - Position

Event occurs at position (x, y, z) and time t is reference frame S.
Same event space-time co-ordinates in S' (left):
Inverse transformation $S' \rightarrow S$ (right):

$$
\begin{aligned}
x' &= \gamma(x - vt) & \qquad x &= \gamma(x' + vt') \\
y' &= y & y &= y' \\
z' &= z & z &= z' \\
t' &= \gamma\left(t - \frac{vx}{c^2}\right) & t &= \gamma\left(t' + \frac{vx'}{c^2}\right)
\end{aligned}
$$

$$\textit{with} \;\; \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Relativity of Simultaneity

Even the temporal order of events is relative!

- Two spatially separated events simultaneous in one reference frame are not simultaneous in any other frame moving relative to the first.
- Clocks synchronized in one frame are not syncronized in any other frame.

Many so-called paradoxes are resolved from careful consideration of relativity of simultaneity.

Lorentz Transformations - Velocity

$$
\begin{aligned}
u'_x &= \frac{u_x - v}{1 - \frac{vu_x}{c^2}} \\
u'_y &= \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \\
u'_z &= \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}
\end{aligned}
$$

$$
\begin{aligned}
u_x &= \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}} \\
u_y &= \frac{u'_y}{\gamma\left(1 + \frac{vu'_x}{c^2}\right)} \\
u_z &= \frac{u'_z}{\gamma\left(1 + \frac{vu'_x}{c^2}\right)}
\end{aligned}
$$

Time Dilation and Length Contraction

- Time dilation:

$$T = \gamma T'$$

- Length contraction:

$$L = \frac{L'}{\gamma}$$

Time Dilation Example

Time dilation example: muon decay.

Muons have an average lifetime of $2.2\mu s$. Muons traveling at 0.99c can travel about 650m before decaying. Therefore almost all muons produced in the upper atmosphere will decay before they reach the earth ....or will they??

Relative to the earth (lab frame):

$$
\begin{aligned}
t &= \frac{t'}{\sqrt{1-\frac{v^2}{c^2}}} \\
t &= \frac{2.2\mu s}{\sqrt{1-0.99^2}} \\
t &= 16\mu s
\end{aligned}
$$

$\Longrightarrow$ In lab frame muon can travel ~4.7km

From the muon's point of view it is instead length contraction which saves the day.

Four-Vectors

We define a position-time 4-vector as

$$x^0 = ct, x^1 = x, x^2 = y, x^3 = z$$

The Lorentz Transformations are then

$$\begin{aligned} x^{0\prime} &= \gamma(x^0 - \beta x^1) \\ x^{1\prime} &= \gamma(x^1 - \beta x^0) \\ x^{2\prime} &= x^2 \\ x^{3\prime} &= x^3 \end{aligned}$$

where $\beta = v/c$.

Four-Vectors

This can be written as

$$x^{\mu\prime} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} x^{\nu}, (\mu = 0, 1, 2, 3)$$

The coefficients are

$$\begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Using Einstein's summation convention the Lorentz transformations are:

$$x^{\mu\prime} = \Lambda^{\mu}_{\nu} x^{\nu}$$

A **four-vector** is a four-component object which behaves like x^{μ} under Lorentz transformations.

A Lorentz-Invariant Quantity

There is a combination of event coordinates which is Lorentz invariant:

$$I = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x^{0\prime})^2 - (x^{1\prime})^2 - (x^{2\prime})^2 - (x^{3\prime})^2$$

It would be nice to write these as a sum but for the minus signs.
Introduce the metric $g_{\mu\nu}$:

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The invariant is now:

$$I = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} x^\mu x^\nu = g_{\mu\nu} x^\mu x^\nu$$

A Lorentz-Invariant Quantity

We have defined the contravariant 4-vector x^μ. We can also define the covariant 4-vector

$$x_\mu \equiv g_{\mu\nu} x^\nu$$

In other words

$$x_0 = x^0, x_1 = -x^1, x_2 = -x^2, x_3 = -x^3$$

We can then write the invariant as

$$I = x_\mu x^\mu$$

This is like a 4-dimensional dot-product.
Notations:

- $x \cdot x$ means $x_\mu x^\mu$
- 4-vectors are written in small letters "x" & 3-vectors are written is bold letters "**x**"

Vectors and Tensors

- A second rank tensor $s^{\mu\nu}$ carries two indices, has $4^2 = 16$ components and transforms with two factors of Λ:

$$s^{\mu\nu\prime} = \Lambda^{\mu}_{\kappa}\Lambda^{\nu}_{\sigma}s^{\kappa\sigma}$$

- A third rank tensor $t^{\mu\nu\lambda}$ carries three indices, has $4^3 = 64$ components and transforms with three factors of Λ:

$$t^{\mu\nu\lambda\prime} = \Lambda^{\mu}_{\kappa}\Lambda^{\nu}_{\sigma}\Lambda^{\lambda}_{\tau}t^{\kappa\sigma\tau}$$

- A **four-vector** is a four-component object which behaves like x^μ under Lorentz transformations.
  Vector is a tensor of rank one.
- A scalar is a tensor of rank zero.

Conserved vs. Invariant quantities

- A **conserved** quantity remains the same, **in a particular frame**, before and after an event.
- An **invariant** quantity is the same in **all inertial reference frames**.

- Energy is conserved, but not invariant.
- Mass is invariant, but not conserved.

Collisions

- Classically, we always conserve 3-momentum (**p**), always conserve mass, sometimes conserve kinetic energy, and always conserve total energy even if we dont keep track of it all.

- Relativistically, we always conserve 3-momentum (**p**), sometimes conserve mass, sometimes conserve kinetic energy, and always conserve total energy.

  More succinctly, four-momentum is conserved.

$$p_A^\mu + p_B^\mu = p_C^\mu + p_D^\mu$$

for a collision of $A + B \rightarrow C + D$

Energy and Momentum

Because proper time is invariant under Lorentz transformations it is convenient to use in particle physics. So, Griffiths defines a proper velocity:

$$\boldsymbol{\eta} = \frac{d\boldsymbol{x}}{d\tau}$$

which is related to the lab velocity by

$$\boldsymbol{\eta} = \gamma \boldsymbol{v}$$

η is actually part of a 4-vector

$$\eta^\nu = \frac{dx^\mu}{d\tau} = \gamma(c, v_x, v_y, v_z)$$

Energy and Momentum

Why does Griffiths define this funny η thing? It is because he wants to rewrite classical momentum in a relativistically correct way by just swapping-out the velocity

$$\boldsymbol{p} = m\boldsymbol{\eta}$$

Proper velocity is part of a 4-vector and so is momentum

$$p^\mu = m\eta^\mu$$

and

$$p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right)$$

where E is defined in the relativistically correct way:

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

Energy and Momentum

Note that in classical mechanics there is no such thing as a massless particle but now we have

$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - v^2/c^2}}$$

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

Notice what happens when $m = 0$ and $v = c$!
In the massless case we have

$$E = |\boldsymbol{p}|c$$

Table 1.1. *Units in high energy physics*

(a)

Quantity	High energy unit	Value in SI units
length	1 fm	10^{-15} m
energy	1 GeV = 10^9 eV	1.602×10^{-10} J
mass, E/c^2	1 GeV/c^2	1.78×10^{-27} kg
$\hbar = h/(2\pi)$	6.588×10^{-25} GeV s	1.055×10^{-34} J s
c	2.998×10^{23} fm s^{-1}	2.998×10^8 m s^{-1}
$\hbar c$	0.1975 GeV fm	3.162×10^{-26} J m

(b)

natural units, $\hbar = c = 1$		
mass, Mc^2/c^2	1 GeV	
length, $\hbar c/(Mc^2)$	1 GeV^{-1} = 0.1975 fm	
time, $\hbar c/(Mc^3)$	1 GeV^{-1} = 6.59×10^{-25} s	
Heaviside–Lorentz units, $\epsilon_0 = \mu_0 = \hbar = c = 1$		
fine structure constant	$\alpha = e^2/(4\pi) = 1/137.06$	
Relations between energy units		
1 MeV = 10^6 eV	1 GeV = 10^3 MeV	1 TeV = 10^3 GeV

Scales in Natural & SI Units to remember

- mass of a proton/neutron $\sim 1 GeV$
- mass of an electron $\sim 0.5 MeV$
- proton/neutron radius $\sim 1 fm = 10^{-15} m = 1 fermi$
- time for light to cross a proton $\sim 10^{-23} sec$
  Time scale of strong interaction.

Energy and Momentum

To sum it up, using Natural units (setting $c = 1$):

- Relativistic Energy is:

$$E = \gamma m$$

- Relativistic momentum is:

$$\boldsymbol{p} = \gamma m \boldsymbol{v}$$

- Define the four-momentum $p^\mu = (E, \boldsymbol{p})$.
  Then, $p^2 = m^2$ is relativistically invariant.
- For massless particles:

$$E = |\boldsymbol{p}| = h\nu$$

Lab and CM frames

Experimental setups for carrying out scattering experiments are of two types:
(1) fixed target, where a beam of particles strikes particles at rest and
(2) colliding beams, where counterrotating beam particles strike one another head on.

Two standard Lorentz frames, corresponding to each of these setups, are in wide use.
(1) The LAB frame is the coordinate system in which the beam particle is moving and the target is at rest.
(2) The Center of Momentum (CM) frame is the coordinate system in which the net momentum is zero, i.e., in which the beam and target particle move with equal and opposite momenta.

BaBar

BaBar is an experiment at SLAC which makes use of colliding beams of electrons and positrons: 9GeV e- and 3.1GeV e+.
(a) What are the speeds of the colliding particles?
(b) What are the energies of the particles in the centre-of-mass frame?

We know that $E = \gamma m$ and $m = 0.511 MeV$ (in natural units) so can calculate $\gamma_+ = 6070$ and $\gamma_- = 17600$ and

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} \approx 1 - \frac{1}{2\gamma^2}$$

Giving

$$v_- = (1 - 10^{-9}), v_+ = (1 - 10^{-8})$$

- In the CM frame: $(p_e-) = (E_{CM}, \boldsymbol{p}_{CM})$, $(p_e+) = (E_{CM}, -\boldsymbol{p}_{CM})$ and

$$\begin{aligned}(p_{e-} + p_{e+})^2 &= (2E_{CM}, \mathbf{0})^2 \\ &= 4E_{CM}^2\end{aligned}$$

- In the lab frame: $p_{e-} = (E_-, \boldsymbol{p}_-)$, $p_{e+} = (E_+, \boldsymbol{p}_+)$ and

$$\begin{aligned}(p_{e-} + p_{e+})^2 &= p_{e-}^2 + p_{e+}^2 + 2p_{e-} \cdot p_{e+} \\ &= m^2 + m^2 + 2(E_-E_+ - \boldsymbol{p}_- \cdot \boldsymbol{p}_+) \\ &\approx 2(E_-E_+ + |\boldsymbol{p}_-||\boldsymbol{p}_+|) \\ &\approx 4E_-E_+\end{aligned}$$

- Now equate the CM and lab expressions and get

$$E_{CM} = \sqrt{E_-E_+} = \sqrt{(9GeV)(3.1GeV)} = 5.3GeV$$

Fixed Targets vs Colliding Beams

- So the beam energy is 9 + 3.1 = 12.1 GeV which leads to 10.6GeV CM.
- What if it were a fixed target experiment?
- Use the total four-momentum as an invariant. The individual four-momenta will be $(m, 0)$ and (E, p):

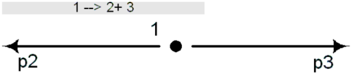
$$
\begin{aligned}
(p_e - +p_e+)^2 &= m^2 + m^2 + 2((m, \mathbf{0}) \cdot (E, \mathbf{p})) \\
&\approx 2Em
\end{aligned}
$$

so, $2Em = 4E_{CM}^2$ and $E = 10^5 GeV$!

- Incredibly large fixed target energies would be required to reach the energies in colliding beam experiment.

Two-body decay

- Consider $\pi \to \mu + \nu$ In the CM frame the two products emerge back-to-back to conserve momentum.



- Four-vectors:

$$p_\pi = (m_\pi, 0), p_\mu = (E_\mu, \mathbf{p}), p_\nu = (E_\nu, \textbf{-p})$$

- Conservation of 4-momentum:

$$p_\pi = p_\mu + p_\nu \implies p_\mu = p_\pi - p_\nu$$

$$\begin{aligned} p_\mu^2 &= (p_\pi - p_\nu)^2 \\ m_\mu^2 &= p_\pi^2 + p_\nu^2 - 2p_\pi \cdot p_\nu = m_\pi^2 - 2m_\pi E_\nu \\ \implies E_\nu &= \frac{m_\pi^2 - m_\mu^2}{2m_\pi} \end{aligned}$$

Two-body decay

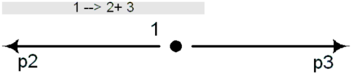
- Similarly $p_\nu = p_\pi - p_\mu$

$$\begin{aligned} p_\nu^2 &= (p_\pi - p_\mu)^2 \\ 0 &= p_\pi^2 + p_\mu^2 - 2p_\pi \cdot p_\mu = m_\pi^2 + m_\mu^2 - 2m_\pi E_\mu \\ \Longrightarrow E_\mu &= \frac{m_\pi^2 + m_\mu^2}{2m_\pi} \end{aligned}$$

- Notice that

$$E_\nu + E_\mu = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} + \frac{m_\pi^2 + m_\mu^2}{2m_\pi} = m_\pi$$

as we require energy conservation.



Three-body decay

- Consider decays such as $n \rightarrow p + e + \bar{\nu}_e$ and $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$
- In the CM frame, the final-state energies are **not** unique.
- The observation that there was a range of electron energies in the two decays above played a large role in the postulate of the existence of neutrinos.

Maximum allowed energy in 3-body decay involving ν

- Consider the decay $K^0 \rightarrow \pi^- + e^+ + \nu_e$ what is the maximum e^+ momentum?
- Assume e^+ as massless. $E_\nu = |\boldsymbol{p}_\nu|$, $E_e = |\boldsymbol{p}_e|$ and $p_K = (m_K, \boldsymbol{0})$

$$p_\pi^2 = (p_K - p_e - p_\nu)^2$$
$$m_\pi^2 = m_K^2 - 2m_K(E_e + E_\nu) + 2E_eE_\nu(1 - cos\theta_{e\nu})$$
$$E_e = \frac{m_K^2 - m_\pi^2 - 2m_K E_\nu}{2(m_K - E_\nu(1 - cos\theta_{e\nu}))}$$

E_e is maximized when $cos\theta_{e\nu} = -1$ and $E_\nu = 0$

$$E_e^{max} = \frac{m_K^2 - m_\pi^2}{2m_K} = 229MeV$$

- We get the same result when we set $E_\nu = 0$. More generally:

$$E_e^{max} = \frac{m_K^2 + m_e^2 - m_\pi^2}{2m_K}$$

Invariant mass

- Consider particle X decay: $X \to e^{-} + e^{+}$
- Combine measurements of energy and momenta of electron-positron pairs to make the four-momentum-squared invariant (square root of which is known as an **invariant mass**).

$$m_{inv}^2 = (E_{-} + E_{+})^2 - (\boldsymbol{p}_{-} + \boldsymbol{p}_{+})^2$$

- Since energy and momentum are conserved:

$$E_X = E_{-} + E_{+} \;\; and \;\; \boldsymbol{p}_{\boldsymbol{X}} = \boldsymbol{p}_{-} + \boldsymbol{p}_{+}$$

- Therefore, the four-momentum-squared invariant built for electron-positron pairs, would give the mass of the X-particle:

$$m_{inv}(pair) = m_X$$

Rapidity and Pseudorapidity

Another very frequently used invariant is known as rapidity.
It is used instead of polar angle (θ) in hadron collider experiments because the CM frame of the interacting particles is boosted in the lab at hadron colliders. It is defined as

$$y = \frac{1}{2}\ln\left(\frac{E+p_z}{E-p_z}\right) = \tanh^{-1}\left(\frac{p_z}{E}\right)$$

The really neat feature of this variable is that a Lorentz transformation gives:

$$y - \tanh^{-1}\beta$$

the derivative of which (wrt y) is the same as the derivative of y itself. So, the shape of the rapidity distribution is Lorentz invariant!

Rapidity and Pseudorapidity

In the limit in which $p >> m$ (high energy particles) you can define pseudorapidity as

$$\eta = -\ln \tan(\frac{\theta}{2}) \approx y$$

Notice that this quantity does not depend on the mass or energy of the particle, it is just a function of the polar angle.